

## Coherent structures in quantum systems with two coupled types of excitation

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This paper investigates the consequences of coupling two different types of quantum quasiparticle in many-body systems. It utilizes a nonlinear field theoretical approach. Four main cases have been studied depending on the statistics of the component quasiparticles and the attendant equations of motion for the quantum fields have been derived. In each case Poincaré sections reveal the presence of chaotic as well as regular behavior in the phase space. Within the class of regular orbits our interest is focused on separatrices and the appearance and disappearance of periodic solutions. The general method is illustrated by two examples: (a) distinguishable fermion particles and (b) an electron-phonon system. In the latter case we derive criteria for the emergence of localized states. [S1063-651X(98)07603-X]

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### I. INTRODUCTION

A large variety of condensed matter systems exhibit properties which manifest underlying competition between two distinct types of order or degrees of freedom. Examples of such behavior are numerous within the condensed matter physics area and include metamagnets [1], ferroelectric-ferromagnetic systems [2], ferroelectric-piezoelectric crystals [3], crystalline-superfluid systems [4], as well as orientation-position ordering phenomena in molecular liquid crystals [4], to name just a few. Two different types of order may engage in a competition, e.g., superconductivity and ferromagnetism, ferromagnetism and antiferromagnetism [1], ferromagnetism and ferroelectricity [2]; different lattice deformation components give rise to a plethora of ferroelastic behaviors [5] and even two-mode lasers should be considered to be examples of this complex behavior. It is well known that an interplay between two distinct orders may result in critical temperature shifts as well as crossover phenomena. This can be readily analyzed using the mean field approximation [5]. In particular, Schulman [6] used catastrophe theory to predict the effects of the coupling on the relevant phase diagram for systems with several order parameters. Specific applications to Landau-type expansions of free energies with two coupled order parameters were made [7,8a] and simple rules were derived for multicritical behavior of the systems involved. Another situation which is quite prevalent in many-body physics involves the coupling of order parameters (critical degrees of freedom) to nonordering variables such as elastic variables [8b] or electromagnetic fields [8c] to, for example, spin degrees of freedom. These couplings can lead to a rich variety of possible behaviors including critical crossover effects [8d] creation of Wigner-order critical points [8e] and, in addition to these equilibrium effects, dynamic phenomena such as instabilities or solitons. Imry [8f] pointed out that in view of the universal behaviors

classified by renormalization group theory, appropriate models to study such coupled-fields phenomena would just need to be field-theory Hamiltonians in generalized Landau-Ginzburg form. Some twenty years after this proposal has been made we intend to demonstrate in the present paper how a connection can be made between microscopic second-quantized Hamiltonians for coupled systems and Landau-Ginzburg order parameter expansions. Furthermore, we intend to show how in practice the resultant equations of motion for coupled order parameter fields may be analyzed and, in many cases, exactly solved. Concisely, our intention is to provide the reader with a practical toolbox for dealing with coupled quantum many-body systems close to criticality in one or both variables. Our emphasis will be placed on extracting nonlinear features of the dynamic behavior with such attendant properties as solitons, localization, and chaotic regions of the phase space.

As realized early in [8f], a more fundamental microscopic approach to the problem poses a serious difficulty due to inherent nonlinearities in the description. The objective of the present paper is to provide an insight into the problem of coupled degrees of freedom using a recently developed field theoretical technique [9,10], which applies to systems composed of a large number of strongly interacting particles. The original method, henceforth referred to as the method of coherent structures (MCS) [9,10], is based on a nonlinear analysis of collective modes of behavior in these many-body systems.

The starting point is the generic second quantized Hamiltonian  $H$  of the form

$$H = \sum_{\mathbf{k}} \omega_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + \sum_{\mathbf{k}, \mathbf{l}, \mathbf{m}} \Delta_{\mathbf{k}, \mathbf{l}, \mathbf{m}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{l}}^{\dagger} a_{\mathbf{m}} a_{\mathbf{k} + \mathbf{l} - \mathbf{m}} \quad (1.1)$$

which includes both one- and two-body interaction terms. This type of Hamiltonian appears in numerous cases in phys-

ics such as electrons in metals, superfluidity, the BCS theory of superconductivity, anharmonic phonons in crystals, etc. In each case the annihilators and creators refer to the annihilation or creation of quasiparticles which may obey either Bose or Fermi-Dirac commutation rules and, in nearly all cases in the plane wave representation, the interaction takes the linear momentum conserving form as in Eq. (1.1). The first step of the method is to derive the equations of motion for a particular annihilator,  $a_{\boldsymbol{\eta}}(t)$  say, using Heisenberg's equation of motion

$$i\hbar\partial_t a_{\boldsymbol{\eta}} = -[H, a_{\boldsymbol{\eta}}]_{-}. \quad (1.2)$$

The commutator is readily calculated using Bose or Fermion commutation rules and in either case may be cast in the same form, i.e.,

$$i\hbar\partial_t a_{\boldsymbol{\eta}} = \omega_{\boldsymbol{\eta}} a_{\boldsymbol{\eta}} + 2 \sum_{\mathbf{k}, \mathbf{m}} \Delta_{\boldsymbol{\eta}, \mathbf{k}, \mathbf{m}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{m}} a_{\mathbf{k} + \boldsymbol{\eta} - \mathbf{m}}. \quad (1.3)$$

In a standard way the one-body and two-body matrix elements,  $\omega_{\boldsymbol{\eta}}$  and  $\Delta_{\boldsymbol{\eta}, \mathbf{k}, \mathbf{m}}$ , respectively, in Eq. (1.3), are then expanded about a particular point in  $\mathbf{k}$  space,  $(\boldsymbol{\eta}_0, \mathbf{k}_0, \mathbf{m}_0)$  (or quantum number space). Thus, when both sides of Eq. (1.3) are multiplied by  $\exp(-i\boldsymbol{\eta} \cdot \mathbf{r})/\sqrt{\Omega}$  and summed over all  $\boldsymbol{\eta}$ , the equation of motion in Eq. (1.3) may be written entirely in terms of a quantum field  $\psi$  and its derivatives where  $\psi$  is defined by

$$\psi = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{k}} \exp(-i\mathbf{k} \cdot \mathbf{r}) a_{\mathbf{k}}. \quad (1.4)$$

The point  $(\boldsymbol{\eta}_0, \mathbf{k}_0, \mathbf{m}_0)$  is now chosen carefully to be a critical or fixed point of the system under consideration. This may also be viewed as an expansion about the correspondence principle limit. One reason for this is that close to such a point the field  $\psi$  is predominantly classical. Secondly, if this point corresponds to a second order phase transition, renormalization group ideas may be used to truncate the Taylor expansion about  $(\boldsymbol{\eta}_0, \mathbf{k}_0, \mathbf{m}_0)$ . The reason for this is that the equation of motion may be shown to be derivable, as an Euler-Lagrange equation from a Hamiltonian functional. Under quite general conditions on the field treated as a classical function, the Hamiltonian density can be written in a standard  $\phi^n$ -field theory form [11,12]. A particular  $\phi^n$  model is well known to be renormalizable when the number of independent space (or space-time) variables  $N_c$  is given by  $N_c = 2n/(n-2)$ . For a  $\phi^4$  model, the Hamiltonian density is renormalizable for  $N_c = 4$  which means that if an expansion away from  $(\boldsymbol{\eta}_0, \mathbf{k}_0, \mathbf{m}_0)$  is made in the Hamiltonian density only terms which deviate away from it to  $O(k^4)$  need be retained. Thus in the equation of motion only terms up to  $O(k^3)$  need be retained, all higher order contributions merely redressing those below those of  $O(k^3)$ . One finds, after considerable algebra [9,10] that for an isotropic or cubic system, the most general field equation of motion is

$$\begin{aligned} i\hbar\partial_t \psi = & \mu_0 \psi + i\boldsymbol{\mu}_1 \cdot \nabla \psi + \mu_2 \nabla^2 \psi - 2(\nabla \psi^{\dagger}) \psi (\nabla \psi) \\ & + \mu_3 \psi^{\dagger} \psi \psi + i\{\psi^{\dagger} \psi (\boldsymbol{\mu}_4 \cdot \nabla) \psi + \psi^{\dagger} [(\boldsymbol{\mu}_4 \cdot \nabla) \psi] \psi\} \\ & + [(\nabla^2 \psi^{\dagger}) \psi \psi + \psi^{\dagger} \psi \nabla^2 \psi] \end{aligned} \quad (1.5)$$

in the Euclidean case, where  $\mu_0$  and  $\mu_3$  are constant parameters and  $\boldsymbol{\mu}_1$  and  $\boldsymbol{\mu}_4$  are constant vectors.

This classical envelope equation can, for example, be solved with the help of recent mathematical discoveries in the area of nonlinear partial differential equations (PDE's) [13–15]. The new powerful mathematical techniques at our disposal allow us, at the very least, to extract important physical information about the local geometry of the emerging coherent structure resulting from many-body interactions. Quite often also the analytical form of these classical nonlinear fields existing in multidimensional space-time can be determined. The final stage in this procedure (MCS) is an analysis of the stability of the classical analytical solutions. Once the stable classical solutions are found they provide an effective potential in a linear Schrödinger equation for the internal quantum oscillations. As an example of this procedure to a system with spherical symmetry and how the effective potential is provided by the classical field, the reader is referred to Ref. [16]. It should be pointed out in this connection that approaches very similar in spirit were proposed almost simultaneously by other authors (see, e.g., Ref. [17]). In the present paper we will not be concerned with quantum fluctuations because of the closeness to the correspondence limit.

Apart from the general framework, several specific physical applications have been recently worked out with the aim of testing the validity of the method. First, the BCS Hamiltonian for superconductivity has been used and a careful analysis resulted in a remarkable confirmation of earlier standard scaling laws for the superconducting current and energy gap [18]. Another application was concerned with the equilibrium phases of metamagnets, i.e., spin systems with two or more sublattices. In this case an independent analytical support was provided [19] for a numerical form of the phase boundaries between the three possible ground states [20]. Furthermore, new insights have been provided into the Haldane gap problem for quantum Heisenberg spin chains [21] and also the bound states in multielectron atoms have been investigated [16]. Another important observation has been recently made [22] in connection with the presence of spin degrees of freedom. This requires the use of a separate field for each spin component. However, it was rigorously demonstrated that the inclusion of spin does not alter the form of the equations of motion for each individual quantum field and hence the basic results of the calculations remain valid. What is affected, however, by the presence of spin is the magnitude of the nonlinear coupling coefficient when the spin  $S > \frac{1}{2}$ , in which case it becomes multiplied by the spin degeneracy  $(2S+1)$  (for integer spins only). This may also lend support to the approach described since the critical temperature will be spin dependent and, for example, for superfluid  $^3\text{He}$  and  $^4\text{He}$  the transition temperature is much higher in the latter case than in the former [23]. Finally, it has been demonstrated that the form of the nonlinear field equations, provided the field is suitably defined, is exactly the same in

any orthonormal basis of states not just a plane-wave basis as originally used [24].

The motivation for this paper is to extend the MCS beyond a single quasiparticle type and include another set of degrees of freedom (critical or noncritical), which may represent either another subsystem (e.g., a heat reservoir) or a different type of quasiparticle (electrons and phonons, for instance), when the two subsystems are coupled together. This is an *important* step in the direction towards realistic modeling of complex many-body interacting systems. In fact, under close scrutiny nearly all many-body structures are comprised of several distinct subsystems or components. Superconductivity, for example, involves electrons and phonons dynamically coupled via an electron-phonon interaction [25] and the interaction is, in fact, key to the existence of the emergent phenomena, i.e., superconductivity. The general situation described here is indeed ubiquitous in condensed matter physics with, for example, various phonon modes interacting amongst themselves and leading to structural instabilities [26]. Practically speaking, any combination of two types of excitation present in a solid may be described in this manner (electron-plasmon, magnon-phonon, electron-polaron, photon-electron interactions, etc.). This paper, therefore, addresses the significant question of the consequences of physical coupling between the two subsystems. We will demonstrate step by step how these generic many-body problems can be dealt with using modern tools of non-linear analysis.

## II. DERIVING THE EQUATIONS OF MOTION

In this section, the MCS is applied to a general Hamiltonian describing two types of particle. Here, the entire physical system is divided into two parts, denoted A and B, with strong interactions within each of the systems and coupling terms between the two subsystems. The form of the coupling terms in this study is quite general and includes one- and two-quantum exchanges between systems A and B with associated second-quantized operators ( $a, a^\dagger$ ) and ( $b, b^\dagger$ ), respectively. The model Hamiltonian will be taken generically as

$$H = H_a + H_b + H_{ab}, \quad (2.1)$$

where

$$H_a = \sum_{\mathbf{k}, \mathbf{l}} \omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \sum_{\mathbf{k}, \mathbf{l}, \mathbf{m}} \delta_{\mathbf{k}, \mathbf{l}, \mathbf{m}}^{(0)} a_{\mathbf{k}}^\dagger a_{\mathbf{l}}^\dagger a_{\mathbf{m}} a_{\mathbf{k}+\mathbf{l}-\mathbf{m}} + \sum_{\mathbf{k}, \mathbf{l}, \mathbf{m}} \{ \epsilon_{\mathbf{k}, \mathbf{l}, \mathbf{m}} a_{\mathbf{k}}^\dagger a_{\mathbf{l}} a_{\mathbf{k}-\mathbf{l}} + \epsilon_{\mathbf{k}, \mathbf{l}, \mathbf{m}}^* a_{\mathbf{k}-\mathbf{l}}^\dagger a_{\mathbf{l}}^\dagger a_{\mathbf{k}} \}, \quad (2.2)$$

$$H_b = \sum_{\mathbf{k}, \mathbf{l}} \Omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{k}, \mathbf{l}, \mathbf{m}} \Delta_{\mathbf{k}, \mathbf{l}, \mathbf{m}} b_{\mathbf{k}}^\dagger b_{\mathbf{l}}^\dagger b_{\mathbf{m}} b_{\mathbf{k}+\mathbf{l}-\mathbf{m}} + \sum_{\mathbf{k}, \mathbf{l}, \mathbf{m}} \{ \Lambda_{\mathbf{k}, \mathbf{l}, \mathbf{m}} b_{\mathbf{k}}^\dagger b_{\mathbf{l}} b_{\mathbf{k}-\mathbf{l}} + \Lambda_{\mathbf{k}, \mathbf{l}, \mathbf{m}}^* b_{\mathbf{k}-\mathbf{l}}^\dagger b_{\mathbf{l}}^\dagger b_{\mathbf{k}} \}, \quad (2.3)$$

and

$$H_{ab} = \sum_{\mathbf{k}} \{ \xi_{\mathbf{k}} a_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \xi_{\mathbf{k}}^* b_{\mathbf{k}}^\dagger a_{\mathbf{k}} \} + \sum_{\mathbf{k}, \mathbf{l}} \{ \eta_{\mathbf{k}, \mathbf{l}} a_{\mathbf{k}}^\dagger a_{\mathbf{l}} b_{\mathbf{k}-\mathbf{l}} + \eta_{\mathbf{k}, \mathbf{l}}^* b_{\mathbf{k}-\mathbf{l}}^\dagger a_{\mathbf{l}}^\dagger a_{\mathbf{k}} \} + \sum_{\mathbf{k}, \mathbf{l}, \mathbf{m}} \{ \mu_{\mathbf{k}, \mathbf{l}, \mathbf{m}} b_{\mathbf{k}}^\dagger a_{\mathbf{l}}^\dagger a_{\mathbf{m}} a_{\mathbf{k}+\mathbf{l}-\mathbf{m}} + \mu_{\mathbf{k}, \mathbf{l}, \mathbf{m}}^* a_{\mathbf{k}+\mathbf{l}-\mathbf{m}}^\dagger a_{\mathbf{m}}^\dagger a_{\mathbf{l}} b_{\mathbf{k}} \} + \sum_{\mathbf{k}, \mathbf{l}, \mathbf{m}} \{ \gamma_{\mathbf{k}, \mathbf{l}, \mathbf{m}} a_{\mathbf{k}}^\dagger b_{\mathbf{l}}^\dagger b_{\mathbf{m}} b_{\mathbf{k}+\mathbf{l}-\mathbf{m}} + \gamma_{\mathbf{k}, \mathbf{l}, \mathbf{m}}^* b_{\mathbf{k}+\mathbf{l}-\mathbf{m}}^\dagger b_{\mathbf{l}}^\dagger b_{\mathbf{m}}^\dagger a_{\mathbf{k}} \} + \sum_{\mathbf{k}, \mathbf{l}, \mathbf{m}} \{ \alpha_{\mathbf{k}, \mathbf{l}, \mathbf{m}} a_{\mathbf{k}}^\dagger b_{\mathbf{l}}^\dagger b_{\mathbf{m}} a_{\mathbf{k}+\mathbf{l}-\mathbf{m}} + \alpha_{\mathbf{k}, \mathbf{l}, \mathbf{m}}^* a_{\mathbf{k}+\mathbf{l}-\mathbf{m}}^\dagger b_{\mathbf{m}}^\dagger b_{\mathbf{l}} a_{\mathbf{k}} \}, \quad (2.4)$$

where  $H_a$  and  $H_b$  are the associated Hamiltonians for the two separate subsystems A and B, and  $H_{ab}$  is an operator providing interactions between them. Each of the labels  $\mathbf{k}$ ,  $\mathbf{l}$ , and  $\mathbf{m}$  denote a set of quantum numbers for a complete set of states. In view of the fact that spin labels do not alter the form of the MCS equations of motion, provided the interactions themselves are spin independent, and a change of basis also leaves the form of the equations invariant, we drop spin labels and use a plane-wave basis throughout. The interaction terms in  $H_a$  and  $H_b$  are such that linear momentum is a conserved. Thus an assumption about the form of interaction has been made, but this would cover most examples in physics. The terms retained in  $H_a$  and  $H_b$  represent, respectively, a one-particle energy ( $\omega_{\mathbf{k}}, \Omega_{\mathbf{k}}$ ), two-body interactions ( $\delta_{\mathbf{k}\mathbf{l}\mathbf{m}}^{(0)}, \Delta_{\mathbf{k}\mathbf{l}\mathbf{m}}$ ), as well as scattering processes annihilating two quasiparticles and creating one. For phonons, for example, the latter terms would describe umklapp processes. Note that the interaction  $H_{ab}$  has to be Hermitian and this is reflected in the form of Eq. (2.4). Three-body interchanges have been specifically excluded, i.e., six-legged operators will not appear in the Hamiltonian. The reader should not presuppose that in all four cases discussed below, the form of  $H$  [in Eq. (2.1)] is the same but each particular case will never include terms which do not appear above. Thus, when we consider individual cases we can drop terms from Eq. (2.1) as appropriate and do away with the need to have a separate Hamiltonian in each section.

The first step in this procedure is to calculate the Heisenberg equations of motion for the ladder operators of the two systems. In general, four cases can be considered depending on the types of statistics obeyed by the quasiparticles, i.e., (1) boson-boson, (2) indistinguishable fermion-fermion, (3) distinguishable fermion-fermion, and (4) boson-fermion cases. We consider each of these cases in turn as, in general,

they lead to different equations of motion, case three being the simplest, then case four with the boson-boson situation the most complex. For the sake of brevity only the first case will be shown in all detail; the three remaining cases are calculated analogously and thus only the main steps will be given.

*Case 1: the boson-boson system.* For this situation all terms in Eqs. (2.1)–(2.4) are included. The  $a$  and  $b$  operators satisfy Bose-Einstein commutation rules, namely,

$$[a_{\mathbf{k}}, a_{\mathbf{l}}^\dagger]_- = \delta_{\mathbf{k}, \mathbf{l}}, \quad [b_{\mathbf{k}}, b_{\mathbf{l}}^\dagger]_- = \delta_{\mathbf{k}, \mathbf{l}}, \quad (2.5a)$$

$$[a_{\mathbf{k}}, a_{\mathbf{l}}]_- = [a_{\mathbf{k}}^\dagger, a_{\mathbf{l}}^\dagger]_- = [b_{\mathbf{k}}, b_{\mathbf{l}}]_- = [b_{\mathbf{k}}^\dagger, b_{\mathbf{l}}^\dagger]_- = 0, \quad (2.5b)$$

and

$$[a_{\mathbf{k}}, b_{\mathbf{l}}]_- = [a_{\mathbf{k}}, b_{\mathbf{l}}^\dagger]_- = [a_{\mathbf{k}}^\dagger, b_{\mathbf{l}}]_- = [a_{\mathbf{k}}^\dagger, b_{\mathbf{l}}^\dagger]_- = 0. \quad (2.5c)$$

We have specifically excluded interactions or terms internal to a subsystem which create one particle from the vacuum but have allowed transformation of one particle to another and two of one type to scatter off another but in the process linear momentum is conserved. Since there are two-body terms in Eq. (2.4) we include not only interactions which preserve different particles separately but incorporate terms where, for example, two B particles are destroyed, a B particle is created but an A particle also appears to conserve momentum. No processes of the type in which a B particle is destroyed at the expense of creating three A particles have been included. Employing Heisenberg's equation of motion in the form

$$i\hbar \partial_t a_{\mathbf{n}} = [a_{\mathbf{n}}, H]_-, \quad (2.6)$$

we find

$$\begin{aligned} i\hbar \partial_t a_{\mathbf{n}} = & \omega_{\mathbf{n}} a_{\mathbf{n}} + \sum_{\mathbf{l}, \mathbf{m}} (\delta_{\mathbf{n}, \mathbf{l}, \mathbf{m}}^{(0)} + \delta_{\mathbf{l}, \mathbf{n}, \mathbf{m}}^{(0)}) a_{\mathbf{l}}^\dagger a_{\mathbf{m}} a_{\mathbf{n}+\mathbf{l}-\mathbf{m}} + \xi_{\mathbf{n}} b_{\mathbf{n}} \\ & + \sum_{\mathbf{l}} \{ \eta_{\mathbf{n}, \mathbf{l}} a_{\mathbf{l}} b_{\mathbf{n}-\mathbf{l}} + \eta_{\mathbf{l}, \mathbf{n}}^* b_{\mathbf{l}-\mathbf{n}}^\dagger a_{\mathbf{l}} \} + \sum_{\mathbf{l}} \lambda_{\mathbf{n}+\mathbf{l}, \mathbf{l}}^* b_{\mathbf{l}}^\dagger b_{\mathbf{n}+\mathbf{l}} \\ & + \sum_{\mathbf{l}, \mathbf{m}} (\epsilon_{\mathbf{n}, \mathbf{l}, \mathbf{m}} a_{\mathbf{l}} a_{\mathbf{n}-\mathbf{l}} + \epsilon_{\mathbf{n}+\mathbf{l}, \mathbf{l}, \mathbf{m}}^* a_{\mathbf{l}}^\dagger a_{\mathbf{n}+\mathbf{l}} + \epsilon_{\mathbf{l}, \mathbf{n}, \mathbf{m}}^* a_{\mathbf{l}-\mathbf{n}}^\dagger a_{\mathbf{l}}) \\ & + \sum_{\mathbf{l}, \mathbf{m}} \{ \mu_{\mathbf{l}, \mathbf{n}, \mathbf{m}} b_{\mathbf{l}}^\dagger a_{\mathbf{m}} a_{\mathbf{l}+\mathbf{n}-\mathbf{m}} + \mu_{\mathbf{n}+\mathbf{m}-\mathbf{l}, \mathbf{l}, \mathbf{m}}^* a_{\mathbf{m}}^\dagger a_{\mathbf{l}} b_{\mathbf{n}+\mathbf{l}-\mathbf{m}} \\ & + \mu_{\mathbf{l}, \mathbf{m}, \mathbf{n}}^* a_{\mathbf{l}+\mathbf{m}-\mathbf{n}}^\dagger a_{\mathbf{m}} b_{\mathbf{l}} \} + \sum_{\mathbf{l}, \mathbf{m}} \gamma_{\mathbf{n}, \mathbf{l}, \mathbf{m}} b_{\mathbf{l}}^\dagger b_{\mathbf{m}} b_{\mathbf{n}+\mathbf{l}-\mathbf{m}} \\ & + \sum_{\mathbf{l}, \mathbf{m}} \{ \alpha_{\mathbf{n}, \mathbf{l}, \mathbf{m}} b_{\mathbf{l}}^\dagger b_{\mathbf{m}} a_{\mathbf{n}+\mathbf{l}-\mathbf{m}} \\ & + \alpha_{\mathbf{n}+\mathbf{m}-\mathbf{l}, \mathbf{l}, \mathbf{m}}^* b_{\mathbf{m}}^\dagger b_{\mathbf{l}} a_{\mathbf{n}+\mathbf{m}-\mathbf{l}} \} \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} i\hbar \partial_t b_{\xi} = & \Omega_{\xi} b_{\xi} + \sum_{\mathbf{l}, \mathbf{m}} \{ \Delta_{\xi, \mathbf{l}, \mathbf{m}} b_{\mathbf{l}}^\dagger b_{\mathbf{m}} b_{\xi+\mathbf{l}-\mathbf{m}} \\ & + \Delta_{\mathbf{l}, \xi, \mathbf{m}} b_{\mathbf{l}}^\dagger b_{\mathbf{m}} b_{\mathbf{l}+\xi-\mathbf{m}} \} + \xi_{\xi}^* a_{\xi} + \sum_{\mathbf{l}} \eta_{\xi+\mathbf{l}, \mathbf{l}}^* a_{\mathbf{l}}^\dagger a_{\xi+\mathbf{l}} \\ & + \sum_{\mathbf{l}} \{ \lambda_{\xi, \mathbf{l}} b_{\mathbf{l}} a_{\xi-\mathbf{l}} + \lambda_{\mathbf{l}, \xi}^* a_{\mathbf{l}-\xi}^\dagger b_{\mathbf{l}} \} + \sum_{\mathbf{l}, \mathbf{m}} \{ \Lambda_{\xi, \mathbf{l}, \mathbf{m}} b_{\mathbf{l}} b_{\xi-\mathbf{l}} \\ & + \Lambda_{\xi+\mathbf{l}, \mathbf{l}, \mathbf{m}}^* b_{\mathbf{l}}^\dagger b_{\xi+\mathbf{l}} + \Lambda_{\mathbf{l}, \xi, \mathbf{m}}^* b_{\mathbf{l}-\xi}^\dagger b_{\mathbf{l}} \} \\ & + \sum_{\mathbf{l}, \mathbf{m}} \{ \mu_{\xi, \mathbf{l}, \mathbf{m}} a_{\mathbf{l}}^\dagger a_{\mathbf{m}} b_{\xi+\mathbf{l}-\mathbf{m}} \} \\ & + \sum_{\mathbf{l}, \mathbf{m}} \{ \gamma_{\mathbf{l}, \xi, \mathbf{m}} a_{\mathbf{l}}^\dagger b_{\mathbf{m}} b_{\mathbf{l}+\xi-\mathbf{m}} + \gamma_{\xi+\mathbf{m}-\mathbf{l}, \mathbf{l}, \mathbf{m}}^* b_{\mathbf{m}}^\dagger b_{\mathbf{l}} a_{\xi+\mathbf{m}-\mathbf{l}} \\ & + \gamma_{\mathbf{l}, \mathbf{m}, \xi}^* b_{\mathbf{l}+\mathbf{m}-\xi}^\dagger b_{\mathbf{m}} a_{\mathbf{l}} \} + \sum_{\mathbf{l}, \mathbf{m}} \{ \alpha_{\mathbf{l}, \xi, \mathbf{m}} a_{\mathbf{l}}^\dagger b_{\mathbf{m}} b_{\mathbf{l}+\xi-\mathbf{m}} \\ & + \alpha_{\mathbf{l}, \mathbf{m}, \xi}^* a_{\mathbf{l}+\mathbf{m}-\xi}^\dagger b_{\mathbf{m}} a_{\mathbf{l}} \}. \end{aligned} \quad (2.8)$$

In Eqs. (2.7) and (2.8), since the two subsystems are assumed isolated from the environment, momentum conservation requires that the following parameters become diagonal:  $\omega, \Omega, \xi$ .

In order to obtain quantum field equations, two quantum fields are defined corresponding to the two sets of operators as follows

$$\Phi(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} b_{\mathbf{k}} \exp[-i\mathbf{k} \cdot \mathbf{r}], \quad (2.9a)$$

and

$$\psi(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} a_{\mathbf{k}} \exp[-i\mathbf{k} \cdot \mathbf{r}], \quad (2.9b)$$

where  $V$  is a volume over which the plane waves are normalized. For completeness we give the well-known relations

$$\int_v \exp[-i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{r}] d^3 \mathbf{r} = V \delta_{\mathbf{k}_1, \mathbf{k}_2}, \quad (2.10a)$$

$$\sum_{\mathbf{k}} \exp[i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')] = V \delta(\mathbf{r} - \mathbf{r}'). \quad (2.10b)$$

The procedure outlined earlier [9,10] for a single quantum field is now followed and the coefficients in each equation of motion are Taylor expanded about a particular point in reciprocal or quantum number space. It is assumed that there is at least one such ‘point’ which is common to both subsystems. This point may be chosen as an actual critical point of a subsystem with coordinates  $(\mathbf{n}_0, \mathbf{k}_0, \mathbf{m}_0)$  in ‘reciprocal’ space or it may be a point in a regime where the system is close to classical from the correspondence principle [27]. As an example, expanding  $\omega$  we find

$$\omega_{\mathbf{n}} = \omega_{\mathbf{n}_0} + \sum_{s=1}^{\infty} \frac{[(\mathbf{n} - \mathbf{n}_0) \cdot \nabla_{\mathbf{n}}]^s \omega_0}{s!}. \quad (2.11)$$

We expand both  $\omega$  and  $\Omega$  to second order in deviations from the critical point and assume the remaining model parameters to be ‘‘momentum’’ independent. For more details of these expansions see Ref. [9]. Having made the appropriate Taylor expansions and defined fields for each subsystem we multiply both sides of Eqs. (2.7) and (2.8) by  $V^{-1/2} \exp(-i\mathbf{n} \cdot \mathbf{r})$  and  $V^{-1/2} \exp(-i\boldsymbol{\xi} \cdot \mathbf{r})$ , respectively, and then sum over  $\mathbf{n}$  and  $\boldsymbol{\xi}$ . We find the following equation of motion for the quantum field  $\psi$ :

$$\begin{aligned} i\hbar \partial_t \psi &= \omega_0 \psi + i\boldsymbol{\omega}_1 \cdot \nabla \psi + \omega_2 \nabla^2 \psi + 2\delta_0 \psi^\dagger \psi \psi + \xi_0 \Phi \\ &+ \eta_0 \psi \Phi + \eta_0^* \Phi^\dagger \psi + \lambda_0^* \Phi^\dagger \Phi + \epsilon_0 \psi^2 + 2\epsilon_0^* \psi^\dagger \psi \\ &+ \mu_0 \Phi^\dagger \psi^2 + 2\mu_0^* \psi^\dagger \psi \Phi + \gamma_0 \Phi^\dagger \Phi \Phi \\ &+ (\alpha_0 + \alpha_0^*) \Phi^\dagger \Phi \psi, \end{aligned} \quad (2.12)$$

where the new parameters are related in an obvious fashion to those in the original equation, i.e., Eq. (2.7). In a similar manner we obtain, for the other field  $\Phi$ ,

$$\begin{aligned} i\hbar \partial_t \Phi &= \Omega_0 \Phi + i\boldsymbol{\Omega}_1 \cdot \nabla \Phi + \Omega_2 \nabla^2 \Phi + 2\Delta_0 \Phi^\dagger \Phi \Phi + \xi_0^* \psi \\ &+ \eta_0^* \psi^\dagger \psi + \lambda_0 \Phi \psi + \lambda_0^* \psi^\dagger \Phi + \Lambda_0 \Phi^2 + 2\Lambda_0^* \Phi^\dagger \Phi \\ &+ \mu_0 \psi^\dagger \psi \psi + \gamma_0 \psi^\dagger \Phi \Phi + 2\gamma_0^* \Phi^\dagger \Phi \psi \\ &+ (\alpha_0 + \alpha_0^*) \psi^\dagger \Phi \psi. \end{aligned} \quad (2.13)$$

It should be noted that  $\omega_2$  and  $\Omega_2$  in Eqs. (2.12) and (2.13), respectively, are written in this form for convenience since in general they are second order tensors and the Laplacian is replaced by mixed second order derivatives. However, for numerous physical systems they will reduce to a Laplacian operator due to their inherent spatial symmetries. These equations, even when treated classically, are extremely difficult to solve but nevertheless can be analyzed to extract important physical information about the system’s behavior.

*Case 2: fermion-fermion, indistinguishable particles.* As an example of such systems we might imagine two multi-electron atoms in a solid each with their own electrons [28] that may influence each other through mutual interactions. As these particles are indistinguishable, the associated annihilators and creators satisfy the following Fermi-Dirac commutation rules:

$$[a_{\mathbf{k}}, a_{\mathbf{l}}^\dagger]_+ = \delta_{\mathbf{k}, \mathbf{l}}, [a_{\mathbf{k}}, a_{\mathbf{l}}]_+ = [a_{\mathbf{k}}^\dagger, a_{\mathbf{l}}^\dagger]_+ = 0, \quad (2.14a)$$

$$[b_{\mathbf{k}}, b_{\mathbf{l}}^\dagger]_+ = \delta_{\mathbf{k}, \mathbf{l}}, [b_{\mathbf{k}}, b_{\mathbf{l}}]_+ = [b_{\mathbf{k}}^\dagger, b_{\mathbf{l}}^\dagger]_+ = 0, \quad (2.14b)$$

$$[a_{\mathbf{k}}, b_{\mathbf{l}}^\dagger]_+ = [a_{\mathbf{k}}, b_{\mathbf{l}}]_+ = [a_{\mathbf{k}}^\dagger, b_{\mathbf{l}}]_+ = [a_{\mathbf{k}}^\dagger, b_{\mathbf{l}}^\dagger]_+ = 0. \quad (2.14c)$$

Going through the same procedure as before and noting that  $\epsilon$ ,  $\Lambda$ ,  $\eta$ ,  $\lambda$  terms vanish since charge must be conserved, we find for the field equations of motion

$$\begin{aligned} i\hbar \partial_t \psi &= \omega_0 \psi + i\boldsymbol{\omega}_1 \cdot \nabla \psi + \omega_2 \nabla^2 \psi + 2\delta_0 \psi^\dagger \psi \psi + \xi_0 \Phi \\ &+ [(\alpha_0 + \alpha_0^*)] \Phi^\dagger \Phi \psi + \gamma_0 \Phi^\dagger \Phi \Phi - \mu_0 \Phi^\dagger \psi \psi \end{aligned} \quad (2.15)$$

and, for the  $\Phi$  field,

$$\begin{aligned} i\hbar \partial_t \Phi &= \Omega_0 \Phi + i\boldsymbol{\Omega}_1 \cdot \nabla \Phi + \Omega_2 \nabla^2 \Phi + 2\Delta_0 \Phi^\dagger \Phi \Phi + \xi_0^* \psi \\ &- (\alpha_0 + \alpha_0^*) \psi^\dagger \Phi \psi - \gamma_0 \psi^\dagger \Phi \Phi + \mu_0 \psi^\dagger \psi \psi \end{aligned} \quad (2.16)$$

since for Fermions  $\delta_{\mathbf{n}\mathbf{m}}^{(0)} = -\delta_{\mathbf{m}\mathbf{n}}^{(0)}$ . Note that two terms in  $\mu^*$  have cancelled out. This is obvious since otherwise we would either create or destroy a particle.

In Eqs. (2.15) and (2.16), we have again used the scalar symbols  $\omega_2$  and  $\Omega_2$ , respectively, to simplify the second order derivatives so that they take the form of a Laplacian. They can be generalized if need be to give mixed second order derivatives.

*Case 3: fermion-fermion, distinguishable particles.* Here, as an example, we might have a plasma made up of electrons and protons. Unlike case 2, here we must preserve the particle numbers separately for each type so single annihilators or creators for either subsystem are not allowed. The second-quantized operators here satisfy

$$[a_{\mathbf{k}}, a_{\mathbf{l}}^\dagger]_+ = \delta_{\mathbf{k}, \mathbf{l}}, [a_{\mathbf{k}}, a_{\mathbf{l}}]_+ = [a_{\mathbf{k}}^\dagger, a_{\mathbf{l}}^\dagger]_+ = 0, \quad (2.17a)$$

and

$$[b_{\mathbf{k}}, b_{\mathbf{l}}^\dagger]_+ = \delta_{\mathbf{k}, \mathbf{l}}, [b_{\mathbf{k}}, b_{\mathbf{l}}]_+ = [b_{\mathbf{k}}^\dagger, b_{\mathbf{l}}^\dagger]_+ = 0, \quad (2.17b)$$

but between subsystems they commute (not anticommute as in case 2), i.e.,

$$[a_{\mathbf{k}}, b_{\mathbf{l}}^\dagger]_- = [a_{\mathbf{k}}, b_{\mathbf{l}}]_- = [a_{\mathbf{k}}^\dagger, b_{\mathbf{l}}]_- = [a_{\mathbf{k}}^\dagger, b_{\mathbf{l}}^\dagger]_- = 0. \quad (2.17c)$$

For this case  $\epsilon$ ,  $\Lambda$ ,  $\xi$ ,  $\eta$ ,  $\gamma$ ,  $\mu$ ,  $\gamma$ -type interactions are not present. The field equations of motion readily become

$$\begin{aligned} i\hbar \partial_t \psi &= \omega_0 \psi + i\boldsymbol{\omega}_1 \cdot \nabla \psi + \omega_2 \nabla^2 \psi + 2\delta_0 \psi^\dagger \psi \psi \\ &+ (\alpha_0 + \alpha_0^*) \Phi^\dagger \Phi \psi \end{aligned} \quad (2.18)$$

and

$$\begin{aligned} i\hbar \partial_t \Phi &= \Omega_0 \Phi + i\boldsymbol{\Omega}_1 \cdot \nabla \Phi + \Omega_2 \nabla^2 \Phi + 2\Delta_0 \Phi^\dagger \Phi \Phi \\ &+ (\alpha_0 + \alpha_0^*) \psi^\dagger \psi \Phi. \end{aligned} \quad (2.19)$$

They may, at first, appear complicated but much important information can be gained from them and will be analyzed in Sec. III.

*Case 4: fermion (a)-boson (b).* For this particular situation we have in mind the important example of interacting electrons and phonons. Thus  $a$  operators for the Fermions satisfy Fermi-Dirac commutation rules

$$[a_{\mathbf{k}}, a_{\mathbf{l}}^\dagger]_+ = \delta_{\mathbf{k}, \mathbf{l}}, [a_{\mathbf{k}}, a_{\mathbf{l}}]_+ = [a_{\mathbf{k}}^\dagger, a_{\mathbf{l}}^\dagger]_+ = 0, \quad (2.20a)$$

whereas the Boson operators satisfy

$$[b_{\mathbf{k}}, b_{\mathbf{l}}^\dagger]_- = \delta_{\mathbf{k}, \mathbf{l}}, [b_{\mathbf{k}}, b_{\mathbf{l}}]_- = [b_{\mathbf{k}}^\dagger, b_{\mathbf{l}}^\dagger]_- = 0, \quad (2.20b)$$

TABLE I. Painlevé case of the system of two quartic anharmonic oscillators in Eq. (3.4) with  $\mu = \gamma = \epsilon = 0$  following Ref. [29] ( $M$  is the number of arbitrary constants).

Case	Parametric restriction	$M$	Integrals of motion $I_1(=H), I_2$
1(i)	$A=B, \alpha=\beta, \delta=6\alpha$	4	$I_1 = 1/2(p_x^2 + p_y^2) + \alpha(x^4 + y^4 + 6x^2y^2) + A(x^2 + y^2),$
		3	$I_2 = p_x p_y + 2[A + 2\alpha(x^2 + y^2)]xy$
1(ii)	$\alpha=\beta, \delta=2\alpha$	4	$I_1 = 1/2(p_x^2 + p_y^2) + Ax^2 + By^2 + \alpha(x^2 + y^2)^2,$
		3	$I_2 = (xp_y - yp_x)^2 + (2/\alpha)(B-A)[1/2p_x^2 + Ax^2 + \alpha(x^2 + y^2)x^2]$
2b(i)	$A=4B, \alpha=16\beta,$ $\delta=12\beta$	4	$I_1 = 1/2(p_x^2 + p_y^2) + B(4x^2 + y^2) + \beta(16x^4 + y^4 + 12x^2y^2),$
		3	$I_2 = (yp_x - xp_y)p_y + 2(B + 4\beta x^2 + 2\beta y^2)xy^2$
		3	
2b(ii)	$A=4B, \alpha=8\beta, \delta=6\beta$	4	$I_1 = 1/2(p_x^2 + p_y^2) + B(4x^2 + y^2) + \beta(8x^4 + y^4 + 6x^2y^2),$
		3	$I_2 = p_y^4 + 4y^2(B + 6\beta x^2 + \beta y^2)p_y^2 - 16\beta xy^3 p_x p_y$
		3	$+ 4\beta y^4 p_x^2 + 4B^2 y^4 + 4\beta[2B + \beta(2x^2 + y^2)](2x^2 + y^2)y^4$

and between subsystems the operators commute so that

$$[a_{\mathbf{k}}, b_{\mathbf{l}}^\dagger]_- = [a_{\mathbf{k}}, b_{\mathbf{l}}]_- = [a_{\mathbf{k}}^\dagger, b_{\mathbf{l}}]_- = [a_{\mathbf{k}}^\dagger, b_{\mathbf{l}}^\dagger]_- = 0. \quad (2.20c)$$

In this case terms of type  $\epsilon, \xi, \lambda, \mu,$  and  $\nu$  must be omitted from the Hamiltonian. This case is relatively straight forward and we readily derive the equations of motion for the fields as

$$i\hbar \partial_t \psi = \omega_0 \psi + i\boldsymbol{\omega}_1 \cdot \nabla \psi - \omega_2 \nabla^2 \psi + 2\delta_0 \psi^\dagger \psi \psi + \eta_0 \psi \Phi + \eta_0^* \Phi^\dagger \psi + (\alpha_0 + \alpha_0^*) \Phi^\dagger \Phi \psi \quad (2.21)$$

and

$$i\hbar \partial_t \Phi = \Omega_0 \Phi + i\boldsymbol{\Omega}_1 \cdot \nabla \Phi - \Omega_2 \nabla^2 \Phi + 2\Delta_0 \Phi^\dagger \Phi \Phi + \eta_0^* \psi^\dagger \psi + (\alpha_0 + \alpha_0^*) \psi^\dagger \Phi \psi + \Lambda_0 \Phi^2 + 2\Lambda_0^* \Phi^\dagger \Phi. \quad (2.22)$$

### III. ANALYSIS OF THE EQUATIONS AND THEIR SOLUTIONS

In all the cases above the coupled equations of motion are each of the nonlinear Schrödinger (NLS) type with additional terms due to the mutual interactions between the fields. First, there are cross-terms proportional to  $\Phi\Psi$  as well  $|\Phi|^2\Psi$  and  $|\psi|^2\Phi$ . Secondly, ‘‘source’’ terms appear proportional to the other field and its squared modulus. The method for solving these nonlinear coupled differential equations is to first treat the fields as classical and effectively decouple them by assuming a particular type of analytic orbit in the phase space given by the general formula

$$G(\Phi, \Psi) = 0. \quad (3.1)$$

Some commonly used examples of this function involve straight lines, parabolae, ellipses, and hyperbolae [12] but it is not at all guaranteed that any of these approaches will actually produce analytical results in all the cases considered.

Each time a particular orbit is investigated, compatibility conditions have to be satisfied and for stationary orbits the condition

$$\left[ \frac{\partial G}{\partial \Phi} \right]^2 |\nabla \Phi|^2 = \left[ \frac{\partial G}{\partial \Psi} \right]^2 |\nabla \Psi|^2, \quad (3.2)$$

which follows from Eq. (3.1), can be used to effectively decouple the two equations. Rather than provide an extensive general discussion on the applications of this general technique we shall refer the reader to the book by Rajaraman [12] where several cases have been worked out in detail. In principle, depending on the actual form of the systems of coupled equations, a different approach may have to be taken so it is rather hard to provide simple prescriptions for dealing with them.

Limiting our interest to the real and stationary space of solutions maps the problem onto the study of the Hamiltonians of the type

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y), \quad (3.3)$$

where, in general, the potential  $V(x, y)$  is given by

$$V(x, y) = Ax^2 + By^2 + \alpha x^4 + \beta y^4 + \delta x^2 y^2 + \mu xy + \gamma xy^3 + \epsilon yx^3. \quad (3.4)$$

A number of papers have used Painlevé analysis to find conditions for integrability of this type of system [29–34] and a summary of integrable cases is given in Table I. In general, however, the system is not integrable leading to the coexistence of regular and chaotic solutions. For illustration, in Figs. 1–4 we have demonstrated how the shape of the potential  $V(x, y)$  affects the character of the orbits in the phase space by plotting Poincaré sections at different energy levels. In Fig. 1(a) the potential only contains the terms with  $A = B = 1, \alpha = \beta = -1,$  and  $\delta = -1$ . This results in a globally unbounded situation but with a local minimum at  $x = y = 0$ . Consequently, as shown in Fig. 1(b) there is a small region around the origin where regular closed orbits exist. Outside

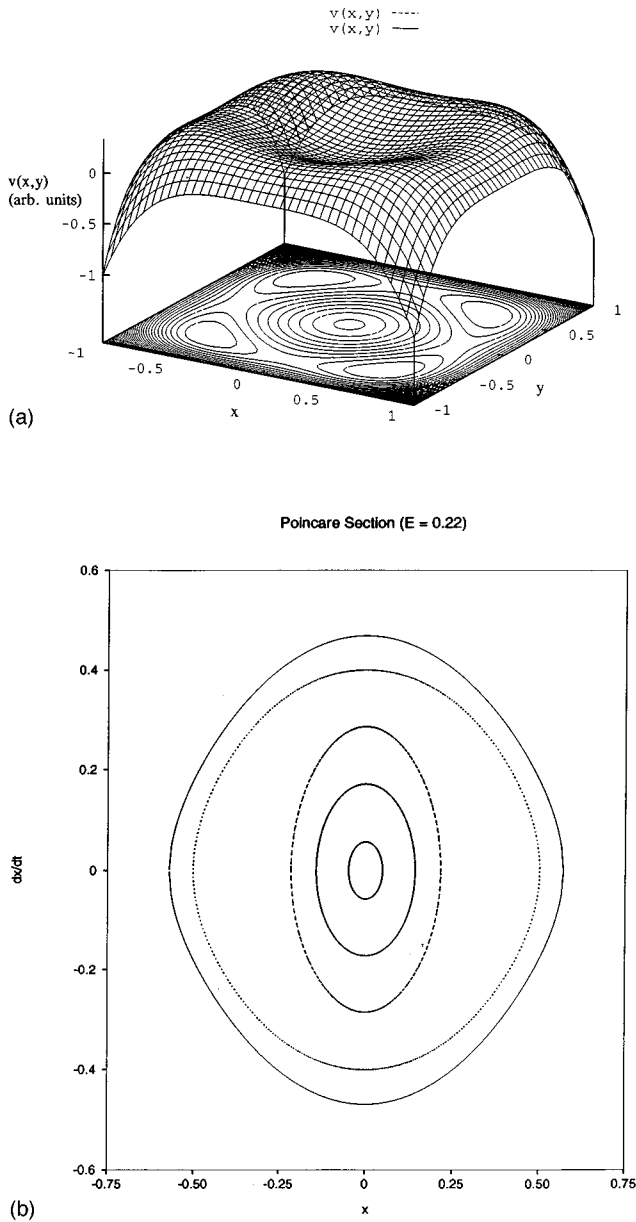


FIG. 1. (a) The potential  $V(x,y)$  of Eq. (3.4) with  $A=B=1$ ,  $\alpha = \beta = -1$ ,  $\delta = -1$ , and the remaining coefficients zero. (b) A Poincaré section  $dx/dt$  versus  $x$  of the corresponding Hamilton's equations for the energy  $E = +0.22$ .

this region there are four maxima but the orbits are unbounded characteristic of singular solutions. In Fig. 2(a) the potential has nonzero terms with  $A=B=-1$ ,  $\alpha = \beta = +1$ , and  $\delta = -1$  such that the potential is asymptotically large and therefore binds the orbits. Figures 2(b)–2(d) illustrate the transition from completely regular to mainly stochastic and eventually completely chaotic behavior as the total energy is gradually raised to correspond to the value of the local maximum at the origin. Figure 3(a) corresponds to  $A = B = 1$ ,  $\alpha = \beta = -1$ ,  $\delta = 1$ . The situation here resembles that in Fig. 1 since there is a finite region where orbits are bounded. However, the presence of three local minima along each diagonal leads to a different topology of the regular orbits in Fig. 3(b). Finally in Fig. 4 we have shown the potential that contains also odd power terms and the coeffi-

icients used are  $A=B=-1$ ,  $\alpha = \beta = \delta = +1$ , and  $\mu = \nu = \epsilon = 1$ . In addition to the familiar elliptic shaped orbits Fig. 4(b) indicates the existence of a separatrix orbit which delineates the boundary between the regular and stochastic region. The latter is fully developed at an energy level corresponding to Fig. 4(c).

A more in-depth analysis of the structure of regular and chaotic solutions to the equations of motion discussed here in general can be made for specific cases. For example, the Fröhlich Hamiltonian for electron-phonon systems is discussed as an illustration in Sec. V. In connection with non-stationary, complex solutions, Malomed [35] considered two coupled nonlinear Schrödinger equations with incoherent and coherent nonlinear couplings and with different group velocities. He showed that two soliton solutions for these equations can form a strongly bound state with a single center of mass or several weakly bound states with far removed centers, provided the coherent type of coupling dominates.

A final comment we wish to make in this section is with respect to the relationship between the MCS and the Landau-Ginzburg theory which makes it very relevant in view of the comments made by Imry [8f] about the formalism required to adequately treat couplings between two degrees of freedom. In the previous paper on the MCS [9] it has been observed that for a single field the equations of motion can be derived via the Euler-Lagrange equations, from a Hamiltonian density of the Landau-Ginzburg form

$$H_{LG}^a = \alpha_a \psi^\dagger \psi + \beta_a \psi^\dagger \psi^\dagger \psi \psi + \gamma_a (\nabla \psi^\dagger) \cdot (\nabla \psi). \quad (3.5)$$

It can be readily shown that the corresponding Hamiltonian density in this problem of two coupled fields can be written in all the cases above within a Landau-Ginzburg formalism as

$$H = H_{LG}^a(\psi, \nabla \psi) + H_{LG}^b(\Phi, \nabla \Phi) + H_{LG}^{ab}(\Phi, \Psi, \nabla \Phi, \nabla \Psi), \quad (3.6)$$

where  $H_{LG}^a$  is that of Eq. (3.5),  $H_{LG}^b$  is obtained from Eq. (3.5) by changing  $\Psi$  to  $\Phi$  and replacing  $a$  by  $b$ . The last term in Eq. (3.6) takes the form (in general)

$$\begin{aligned} H_{LG}^{ab} = & \mu_1 |\Phi|^2 |\Psi|^2 + \mu_2 (\Phi + \Phi^*) |\Psi|^2 + \mu_3 (\Psi + \Psi^*) |\Phi|^2 \\ & + \mu_4 (\psi \Phi^* + \Phi \Psi^*) + \mu_5 (\Phi^* |\Psi|^2 \Psi + \Phi |\Psi|^2 \Psi^*) \\ & + \mu_6 (\Psi^* |\Phi|^2 \Phi + \Psi |\Phi|^2 \Phi^*). \end{aligned} \quad (3.7)$$

On going to first order in the interaction (the reader should consult the original MCS papers Refs. [9,10]) the form of  $H_{LG}^{ab}$  would become modified to include terms which involve a gradient of one field and are proportional to the square or modulus squared of the other field. Second order corrections will bring in Laplacians of one field and squares of the other. In specific systems, cubic terms such as  $(\Phi + \Phi^*) |\Psi|^2$  may be excluded on symmetry grounds but often are important and, for example, in an electron-phonon system,  $\Phi + \Phi^*$  would denote a displacement of the lattice and  $|\Psi|^2$  an electron density distribution. The next section, however, deals with a significantly simpler example of coupled distinguishable Fermion systems. The presence of specific terms above depends on the case considered and for greater clarity of exposition is summarized in Table II. We see from

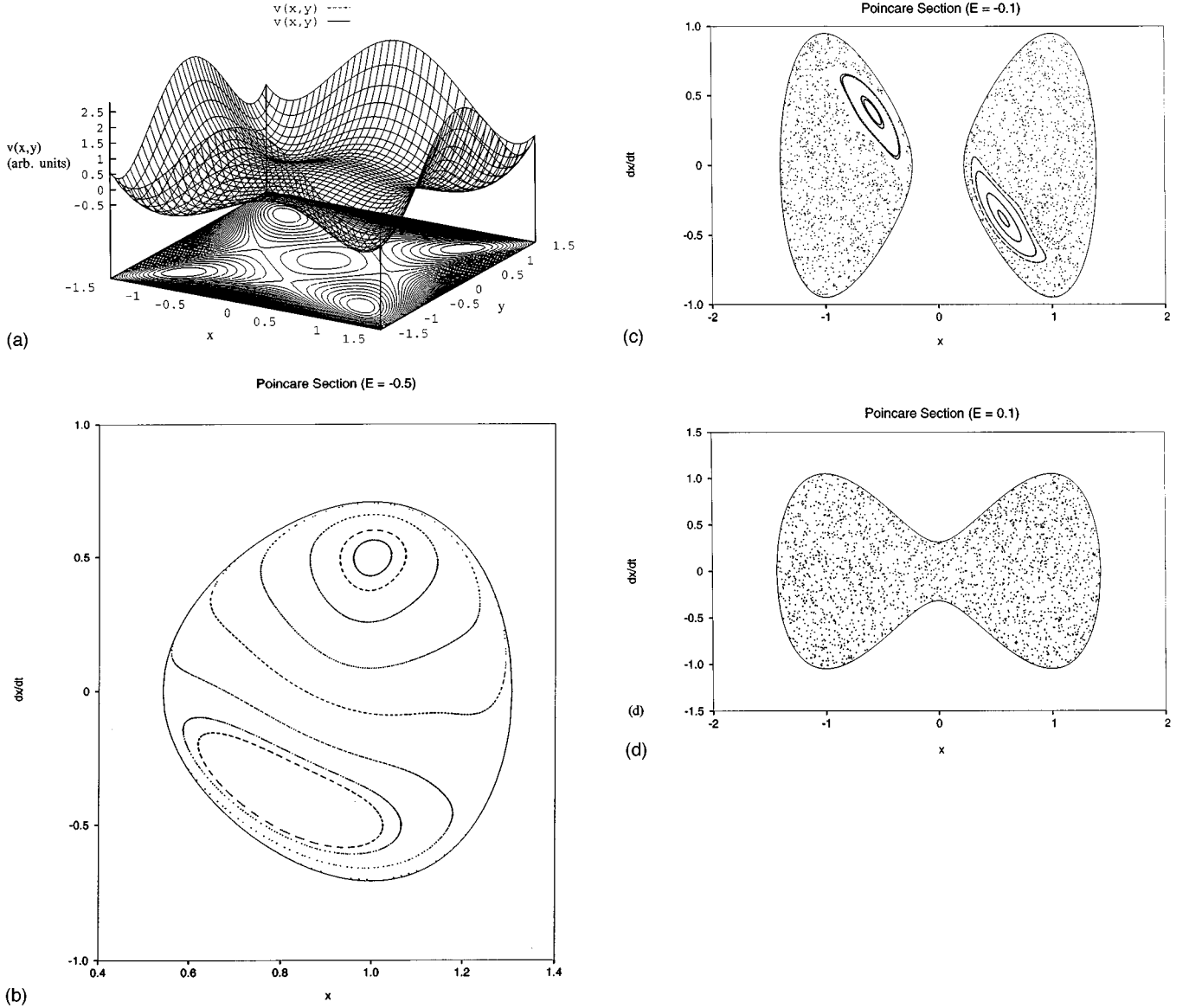


FIG. 2. (a) The potential  $V(x,y)$  of Eq. (3.4) with  $A=B=-1$ ,  $\alpha=\beta+1$ ,  $\delta=-1$ , and the remaining coefficients zero. (b) A Poincaré section  $dx/dt$  versus  $x$  of the corresponding Hamilton's equations for the energy  $E=-0.5$ . (c) Same as in (b) except  $E=-0.1$ . (d) Same as in (b) except  $E=+0.1$ .

this table that the term proportional to  $|\Phi|^2|\Psi|^2$  appears in all cases, the one proportional to  $\Psi\Phi^*$  is only present in the boson-boson and fermion-fermion indistinguishable cases. Terms which are linear in one field and quadratic in the other are only allowed whenever at least one boson field is present. Finally, terms linear in one field and cubic in the other are restricted to the fermion-fermion indistinguishable case.

Before we deal with specific examples presented in the sections that follow we should comment on the rationale for treating the field equations as classical to first order of approximation. In this context, recall that the particle number operator is

$$N = \sum_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} = \int_v d^3x \psi^{\dagger}(x) \psi(x), \quad (3.8)$$

so that  $N$  acting on a state will give as its eigenvalue the total number of particles in the energy level. Thus, treating the

fields  $\psi^+$  and  $\psi$  as classical scalar fields to first order of approximation means that there must be a macroscopically occupied condensate whose density is proportional to  $|\psi|^2$ .

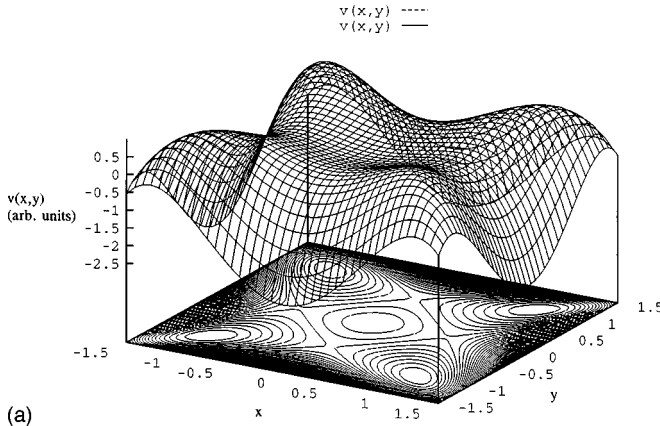
#### IV. SOLUTIONS FOR THE DISTINGUISHABLE FERMION-FERMION CASE (CASE 3)

In this case it is relatively easy to obtain exact solutions to the system of equations of motion for  $\Psi$  and  $\Phi$ , namely, Eqs. (2.18) and (2.19). It is assumed for simplicity that the coordinate system is chosen in such a way that  $\boldsymbol{\omega}_1 \cdot \nabla \psi = \boldsymbol{\Omega}_1 \cdot \nabla \Phi = 0$ . This can be achieved by either a convenient rotation of the coordinates or the choice of a moving frame of reference. The time dependence of the two fields is taken in the form

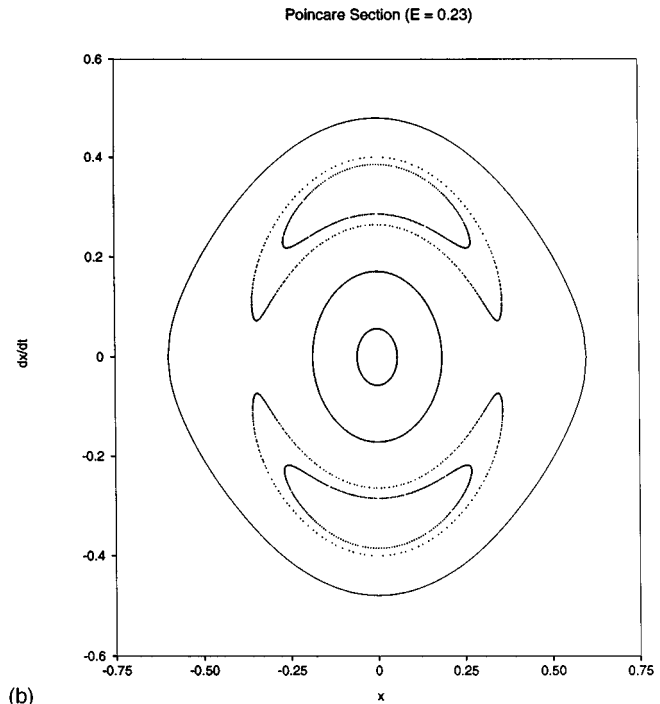
$$\Phi \rightarrow \psi e^{iE_2 t/\hbar}$$

and





(a)



(b)

FIG. 3. (a) The potential  $V(x,y)$  of Eq. (3.4) with  $A=B=+1$ ,  $\alpha=\beta=-1$ ,  $\delta=+1$ , and the remaining coefficients zero. (b) A Poincaré section with  $E=+0.23$ .

$$\psi \rightarrow \psi e^{iE_1 t/\hbar}. \tag{4.1}$$

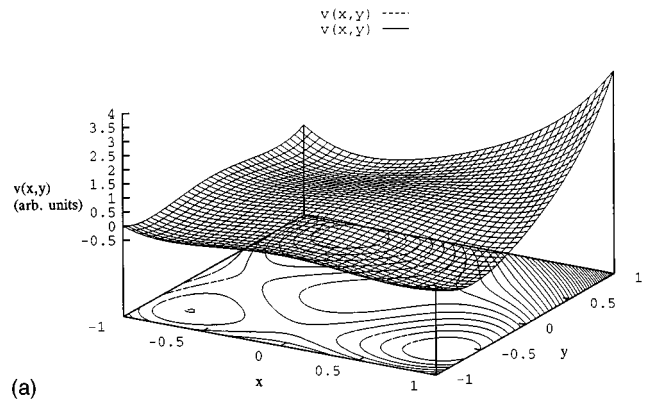
Then, the two fields are assumed to be linearly dependent which yields

$$\Phi = \lambda \psi, \tag{4.2}$$

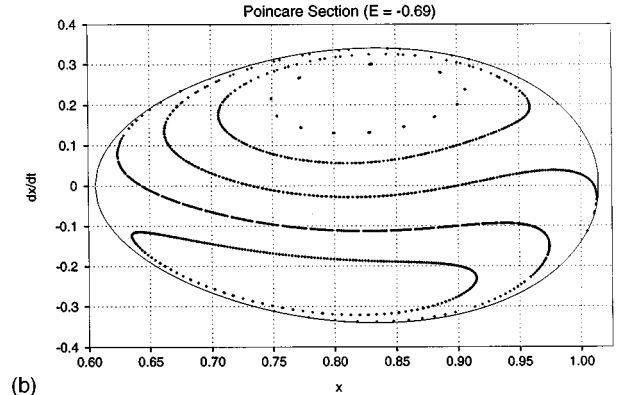
which transforms Eqs. (2.18) and (2.19) into two similar equations. This is certainly the simplest nontrivial type of interdependence but not necessarily the only one allowed by the system. Since the two equations then refer to the same field, compatibility is then required. The equations take the form

$$0 = (\omega_0 + E_1)\psi + \omega_2 \nabla^2 \psi + [2\delta_0 + \lambda^2(\alpha_0 + \alpha_0^*)]\psi^\dagger \psi \psi, \tag{4.3}$$

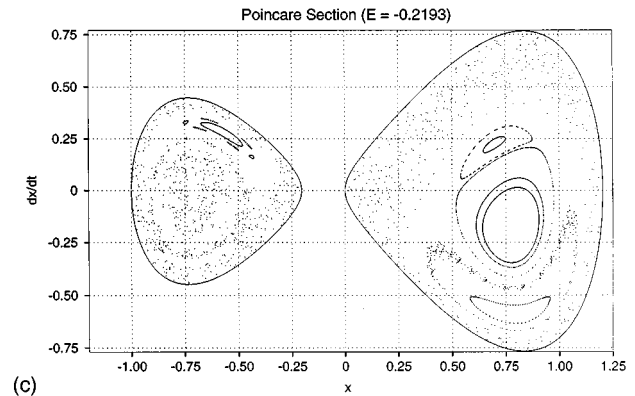
and



(a)



(b)



(c)

FIG. 4. (a) The potential  $V(x,y)$  of Eq. (3.4) with  $A=B=-1$ ,  $\alpha=\beta=+1$ ,  $\delta=+1$ , and  $\mu=\nu=\epsilon=+1$ . (b) A Poincaré section with  $E=-0.69$ . (c) A Poincaré section with  $E=-0.2193$ .

$$0 = (\Omega_0 + E_2)\psi + \Omega_2 \nabla^2 \psi + [2\Delta_0 \lambda^2 + (\alpha_0 + \alpha_0^*)]\psi^\dagger \psi \psi. \tag{4.4}$$

Multiplying Eq. (4.3) on both sides by an arbitrary constant  $\beta$  and comparing corresponding terms in the two equations gives

$$\beta \omega_2 = \Omega_2, \tag{4.5}$$

$$\beta(\omega_0 + E_1) = \Omega_0 + E_2, \tag{4.6}$$

and

$$\beta[2\delta_0 + \lambda^2(\alpha_0 + \alpha_0^*)] = 2\Delta_0 \lambda^2 + (\alpha_0 + \alpha_0^*). \tag{4.7}$$

From Eq. (4.5),  $\beta$  is fixed as  $\beta = \Omega_2/\omega_2$ . Substituting this into Eq. (4.6) we may solve for  $E_2$  to give

TABLE II. The types of coupling terms present for various statistics of quasiparticles.

No.	Case	Nonvanishing two-field interaction potential
1	BB	$\psi\phi^* + \phi\psi^*$ $(\phi + \phi^*) \psi ^2, (\psi + \psi^*) \phi ^2$ $ \phi ^2 \psi ^2$
2	FF Indistinguishable	$\psi\phi^* + \phi\psi^*$ $ \phi ^2 \psi ^2$ $\psi^* \phi ^2\phi + \psi \phi ^2\phi^*$ $\phi^* \psi ^2\psi + \phi \psi ^2\psi^*$
3	FF Distinguishable	$ \phi ^2 \psi ^2$
4	FB	$ \phi ^2 \psi ^2$ $(\psi + \psi^*) \phi ^2, (\phi + \phi^*) \psi ^2$

$$E_2 = \frac{\Omega_2}{\omega_2} (\omega_0 + E_1) - \Omega_0, \quad (4.8)$$

where, of course,  $E_1$  is still arbitrary. Then in Eq. (4.7), substituting for  $\beta$  from Eq. (4.5) we obtain

$$\lambda^2 = \frac{\omega_2/\Omega_2(\alpha_0 + \alpha_0^*) - 2\delta_0}{[(\alpha_0 + \alpha_0^*) - 2(\omega_2/\Omega_2)\Delta_0]}. \quad (4.9)$$

The net result is that the two equations in Eqs. (4.3) and (4.4) are made compatible and both become a stationary *nonlinear Schrödinger equation*. There are a number of solutions of this equation which have been thoroughly investigated in the past. Suffice it to say that, among the spatially inhomogeneous solutions one finds elliptic waves of several kinds as well as hyperbolic localized solutions (solitons) and there exist critical currents above which nonsingular solutions cease to exist [36]. This signifies the onset of dynamical symmetry breaking.

## V. INTERACTING ELECTRONS AND PHONONS

### A. The Fröhlich Hamiltonian

To illustrate the general method described in Sec. II we will use another specific example in which bosons will be represented by phonons and fermions by electrons. Interactions between electrons and phonons in a metal were first described in a quantum mechanical formalism [25] using the Fröhlich Hamiltonian

$$H = \sum_{\mathbf{k}} \frac{\hbar^2}{2m} \mathbf{k}^2 a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \sum_{\mathbf{q}} \hbar \omega_{\mathbf{q}} b_{\mathbf{q}}^\dagger b_{\mathbf{q}} + \sum_{\mathbf{k}, \mathbf{l}} M_{\mathbf{k}, \mathbf{l}} [b_{\mathbf{q}} + b_{-\mathbf{q}}^\dagger] a_{\mathbf{k}}^\dagger a_{\mathbf{l}}, \quad (5.1)$$

where  $\mathbf{q} = \mathbf{k} - \mathbf{l}$ . Here, the operators  $a_{\mathbf{k}}^+$ ,  $a_{\mathbf{k}}$  refer to the electrons while  $b_{\mathbf{q}}^+$ ,  $b_{\mathbf{q}}$  refer to the phonons. This Hamiltonian is apparently one of the simplest possible examples of the general fermion-boson type since only  $\omega$ ,  $\eta$ , and  $\Omega$  terms are retained. Its analysis turns out to be quite involved as we show below. The effective coupling constant  $M_{\mathbf{k}, \mathbf{l}}$  is usually [25] written as

$$M_{\mathbf{k}, \mathbf{l}} = i \sqrt{\frac{N\hbar}{2M\omega_q}} |l - \mathbf{k}| V_{\mathbf{k}-\mathbf{l}}, \quad (5.2)$$

where  $V_{\mathbf{k}-\mathbf{l}}$  is the Fourier transform of a screened Coulomb potential due to a particular ion at the origin and  $N$  is the number of distinct collective coordinates associated with lattice displacements. Typically  $V_{\mathbf{k}-\mathbf{l}}$  takes the form

$$V_{\mathbf{k}-\mathbf{l}} = \frac{2\pi e^2}{|\mathbf{k}-\mathbf{l}|^2 + q_c^2}, \quad (5.3)$$

where  $q_c^{-1}$  is of the order of the interparticle distance and plasma waves only exist when their wavelength is greater than this. In this simple model the ions of the metal are assumed to interact with one another and with conduction electrons via a short-range screened potential. The conduction electrons, on the other hand, are considered to be essentially independent Fermions. Note that the ‘bare’ Coulomb interaction between the ions and the conduction electrons is *not* used, and, to incorporate screening repulsive terms have been built in to some extent via the short-range nature of the effective interaction remaining. The Fröhlich Hamiltonian, in spite of its approximate nature, played an important role in the development of the theory of superconductivity by leading directly to the BCS model.

A fairly up-to-date review of the electron-phonon problem can be found in Ref. [37]. In particular, model parameters for a number of materials can be obtained. However, in this review paper, the theoretical approaches to the problem are predominantly of perturbative type. Our approach emphasizes nonlinearity and nonperturbative effects. This type of procedure for the Fröhlich problems has been studied before. Evans [38] found a macroscopic phonon wavefunction which is due to the displacement field manifesting the off-diagonal long-range order for superconductors. Eremko [39] has recently examined the other side of the problem, i.e., the behavior of electronic charge. He found charge density wave equations of nonlinear Schrödinger type which is consistent with our approach. In what follows, we present the behaviour of both electronic and phonon subsystems within a nonlinear framework of the MCS as outlined in Secs. II and III.

### B. The MCS analysis

We proceed by expanding linearly in  $q^2$  to obtain a  $q$ -dependent correction to the coupling constant. Thus, we obtain

$$\eta_q \cong \eta_0 - \eta_0 \Lambda q^2 + \dots, \quad (5.4)$$

where  $\Lambda = 1/q_c^2$  and

$$\eta_0 = i \sqrt{\frac{N\hbar}{2M\omega(q_F)} \frac{q_F 2\pi e^2}{q_F^2 + q_c^2}} \approx i \sqrt{\frac{N\hbar}{2M|\Omega_2|} \frac{2\pi e^2}{q_c^2}}. \quad (5.5)$$

This will result in the following equations of motion for the  $\Psi$  and  $\Phi$  fields:

$$i\hbar \partial_t \Psi = -\omega_2 \nabla^2 \Psi + 2[\text{Re}(\eta_0 \Phi)] \Psi + 2\Lambda \nabla^2 [\text{Re}(\eta_0 \Phi) \Psi] \quad (5.6)$$

and

$$i\hbar\partial_t\Phi = -\Omega_2\nabla^2\Phi + \eta_0^*|\psi|^2 + \Lambda\eta_0^*\nabla^2(|\Psi|^2). \quad (5.7)$$

To analyze Eq. (5.7) we write

$$\Phi = \Phi_R + i\Phi_I, \quad \eta_0 = i\chi, \quad (5.8)$$

and

$$\Psi = \eta e^{i\phi_0} e^{iEt/\hbar}, \quad (5.9)$$

so that the real and imaginary parts of this equation become, respectively,

$$\text{Re:} \quad \nabla^2\Phi_R = -\frac{\hbar}{\Omega_2}\dot{\Phi}_I, \quad (5.10)$$

$$\text{Im:} \quad \nabla^2(\Omega_2\Phi_I + \Lambda\chi\eta^2) = \hbar\dot{\Phi}_R - \chi\eta^2. \quad (5.11)$$

We then assume that the argument on the left-hand side of Eq. (5.11) is zero so that

$$\Phi_I = -\frac{\Delta\chi\eta^2}{\Omega_2} \quad (5.12)$$

and

$$\dot{\Phi}_R = \frac{\chi\eta^2}{\hbar}. \quad (5.13)$$

In principle, we could add any harmonic function of space and an arbitrary function of time to the right-hand side of Eq. (5.12) so that the left-hand side of Eq. (5.11) is still zero but we draw back at such, possibly unnecessary, complexity. Next we differentiate Eq. (5.10) with respect to time once and substitute Eq. (5.7) for  $\dot{\Phi}_I$  to obtain a wave equation in the form

$$\nabla^2\eta^2 = +\frac{\hbar^2\Lambda}{\Omega_2^2}\partial_{tt}\eta^2, \quad (5.14)$$

the solution of which in one dimension may be written as a wave propagating with velocity  $v$  so that

$$\eta^2 = \eta^2(\xi),$$

where

$$\xi = x - vt \quad (5.15)$$

and  $v = \Omega_2/\hbar\sqrt{\Lambda}$ . Note that the function  $\eta$  in Eq. (5.15) is an *arbitrary* function of  $\xi$  and this does not impose any constraints on the procedure to follow.

We now return to Eq. (5.6) with Eq. (5.12) providing a link and making full use of Eqs. (5.8) and (5.9), to obtain

$$i\hbar\partial_t\Psi = -\omega_2\nabla^2\Psi + 2\frac{\Lambda}{\Omega_2}\chi^2|\Psi|^2\Psi + 2\frac{\Lambda^2}{\Omega_2}\chi^2\nabla^2(|\Psi|^2\Psi). \quad (5.16)$$

This equation will now be solved *exactly*.

In order to interpret  $E$  in the definition of  $\psi$  in Eq. (5.9) we consider the kinetic energy part of the Lagrangian density  $L = T - V$  which takes the form

$$T = \frac{i}{2}(\Psi\Psi_t^* - \Psi^*\Psi_t). \quad (5.17)$$

Thus, inserting Eqs. (5.8) and (5.9) into Eq. (5.17) gives the kinetic energy  $T$  as

$$T = E\frac{\eta^2}{\hbar}. \quad (5.18)$$

Hence, it is clear that  $E$  must be positive since the kinetic energy is proportional to  $E$  and the amplitude squared of the field.

Writing  $\psi$  in modulus-argument form and separating real and imaginary parts yields for Eq. (5.16)

$$\begin{aligned} \text{Re:} \quad -\hbar\eta\frac{\partial\phi_0}{\partial t} + E\eta = & -\omega_2[\nabla^2\eta - \eta(\nabla\phi_0)^2] + \lambda\eta^3 \\ & + \lambda\Lambda[6\eta(\nabla\eta)^2 + 3\eta^2\nabla^2\eta \\ & - \eta^3(\nabla\phi_0)^2], \end{aligned} \quad (5.19)$$

and

$$\begin{aligned} \text{Im:} \quad \hbar\frac{\partial\eta}{\partial t} = & -\omega_2[2\nabla\eta\cdot\nabla\phi_0 + \eta\nabla^2\phi_0] \\ & + \lambda\Lambda\eta^2[6\nabla\eta\cdot\nabla\phi_0 + \eta\nabla^2\phi_0], \end{aligned} \quad (5.20)$$

where  $\lambda \equiv 2\chi^2\Lambda/\Omega_2$ . Equation (5.20) can be understood as a form of *continuity equation* as follows. Defining charge density as  $\rho = \eta^2$  and current density as  $\mathbf{j} = \eta^2\nabla\phi_0$ , it can be written as

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\left[\left(\frac{2\omega_2}{\hbar} - \frac{8\lambda\Lambda\eta^2}{\hbar}\right)\mathbf{j}\right] = 0, \quad (5.21)$$

where the second term in the square bracket of Eq. (5.21) is due to the electron-phonon interaction and is only nonzero in the superconducting state, when  $\eta \neq 0$ . First, a *superconducting current* is proportional to the charge density  $\eta^2$  of the superconducting electrons and acts against the Cooper pair formation, eventually destroying it entirely at a critical current value. Secondly, a *phonon drift current* resulting from electron-phonon coupling (implicitly) acts in the other direction enhancing the growth of the superconducting condensate.

Now, in order to solve Eq. (5.19) we insert  $\mathbf{j} = \eta^2\nabla\phi_0$  and find

$$\begin{aligned} \mathbf{j}\cdot\frac{\hbar\mathbf{v}}{\eta} + E\eta = & -\omega_2\left[\nabla^2\eta - \frac{j^2}{\eta^3}\right] + \lambda\Lambda\left[6\eta(\nabla\eta)^2\right. \\ & \left.+ 3\eta^2\nabla^2\eta - \frac{j^2}{\eta}\right] + \lambda\eta^2. \end{aligned} \quad (5.22)$$

Multiplying through by  $\eta^3$  and introducing, in one-dimension,  $x$ ,

$$\frac{d\eta}{dx} = P$$

and

$$\frac{d^2\eta}{dx^2} = P \frac{dP}{d\eta}, \quad (5.23)$$

we find,

$$[-\omega_2 \eta^3 + 3\lambda\Lambda \eta^5] P \frac{dP}{d\eta} = -P^2 \lambda \Lambda 6 \eta^4 + [\hbar v \eta^2 + E \eta^4 - \lambda \eta^6 - j^2 \omega_2 + \lambda \Lambda \eta^2 j^2]. \quad (5.24)$$

This can be put into the form of a first order linear differential equation by defining

$$Z = P^2, \quad (5.25)$$

to yield

$$\begin{aligned} \frac{dZ}{d\eta} + \frac{12\lambda\Lambda \eta Z}{[\omega_2 + 3\lambda\Lambda \eta^2]} \\ = \frac{2[\hbar v \eta^2 j + E \eta^4 - \lambda \eta^6 - j^2 \omega_2 + \lambda \Lambda \eta^2 j^2]}{[-\omega_2 + 3\lambda\Lambda \eta^2] \eta^3}. \end{aligned} \quad (5.26)$$

This is solved using the integrating factor method as

$$Z = \frac{1}{(-\omega_2 + 3\lambda\Lambda \eta^2)^2} \int \frac{d\eta 2\{\hbar v \eta^2 j + E \eta^4 - \lambda \eta^6 - j^2 \omega_2 + \lambda \Lambda \eta^2 j^2\} (-\omega_2 + 3\lambda\Lambda \eta^2)}{\eta^3}. \quad (5.27)$$

Thus, we can formally write the solution for  $\eta$  in an implicit form as

$$\xi - \xi_0 = \int \frac{d\eta}{\sqrt{Z(\eta)}}. \quad (5.28)$$

As was mentioned earlier, superconductivity corresponds to either the existence of a mean field solution (constant in space and time) or a localized solution which would describe nucleation of superconducting grains. A graphical analysis of Eq. (5.28) provides a quantitative criterion for the latter situation. Note here that localized solutions correspond to the existence of at least two different real roots of  $Z(\eta)$ , one of which *must* be a multiple root.

From Eq. (5.28) the resultant integration constant will determine the values of the corresponding roots of  $Z(\eta)$ . Analyzing the asymptotic behavior of  $Z(\eta)$  for  $\eta \rightarrow \infty$  and  $\eta \rightarrow 0$  we find that in both limits  $Z(\eta) \rightarrow -\infty$ . In searching for localized solutions we must determine whether there exists more than *one* local extremum of  $Z(\eta)$  for  $\eta > 0$ . This leads us to examine the integrand in Eq. (5.28) for  $Z$ . In general, the situation is very complicated but the following asymptotic behavior can be readily found.

Since, for small values of  $\mathbf{j}$ , we will have up to two positive extrema for  $Z(\eta)$ , we will not obtain localized solutions in this regime because a local minimum and a maximum are needed. On the other hand, for large values of  $\mathbf{j}$ , the asymptotic behavior is that  $Z(\eta) \rightarrow 0$  for  $\eta \rightarrow \infty$  and  $Z(\eta) \rightarrow -\infty$  for  $\eta \rightarrow 0$ . This can be made consistent with the existence of a single minimum and a maximum. In Figs. 5–7 we have shown numerical plots of  $Z(\eta)$  for various values of model parameters. We have used the following symbols:

$$\alpha = \frac{\lambda}{E}, \quad \beta = \frac{\lambda\Lambda}{\omega_2}, \quad \gamma = \frac{\lambda\Lambda j^2}{E},$$

and

$$\delta = \frac{\hbar v j}{E}. \quad (5.29)$$

Horizontal bars in Figs. 5–7 denote nonsingular *periodic* solutions corresponding to a highly modulated superconducting state while horizontal dots represent nonsingular *localized* solutions which manifest a nucleating superconducting state. Obviously, a latter possibility is by far more advantageous energetically and, under normal circumstances, this would be chosen as a manifestation of the system's transition to a superconducting phase. From Figs. 5–7 it therefore appears that one should expect a narrow ‘‘window of opportunity’’ in terms of model parameters for the electron-phonon coupling to lead to the formation of a superconducting condensate.

We wish to make a comment with respect to multidimensional solutions. We, therefore, reexamine Eq. (5.19) and set  $(\nabla\phi_0)^2 = \alpha^2$  being time independent. Equation (5.20) can be satisfied by requiring that  $\eta$  and  $\phi$  depend on mutually orthogonal variables. Then, rewriting Eq. (5.19) yields

$$\begin{aligned} (\omega_2 - 3\lambda\Lambda \eta^2) \nabla^2 \eta = (-E + \omega_2 \alpha^2) \eta + 6\lambda\Lambda \eta (\nabla \eta)^2 \\ + (\lambda - \alpha^2 \lambda \Lambda) \eta^3. \end{aligned} \quad (5.30)$$

First of all, Eq. (5.30) may be satisfied by constant solutions given by

$$\eta_0 = 0$$

or

$$\eta_0^2 = \frac{E - \omega_2 \alpha^2}{\lambda(1 - \alpha^2 \Lambda)}. \quad (5.31)$$

and, unlike the one-dimensional case, for positive energies and sufficiently small values of  $\alpha = \nabla\phi$  a *nonzero* order parameter amplitude can always be found. Nonconstant solutions are also possible and Eq. (5.30) can be solved in an

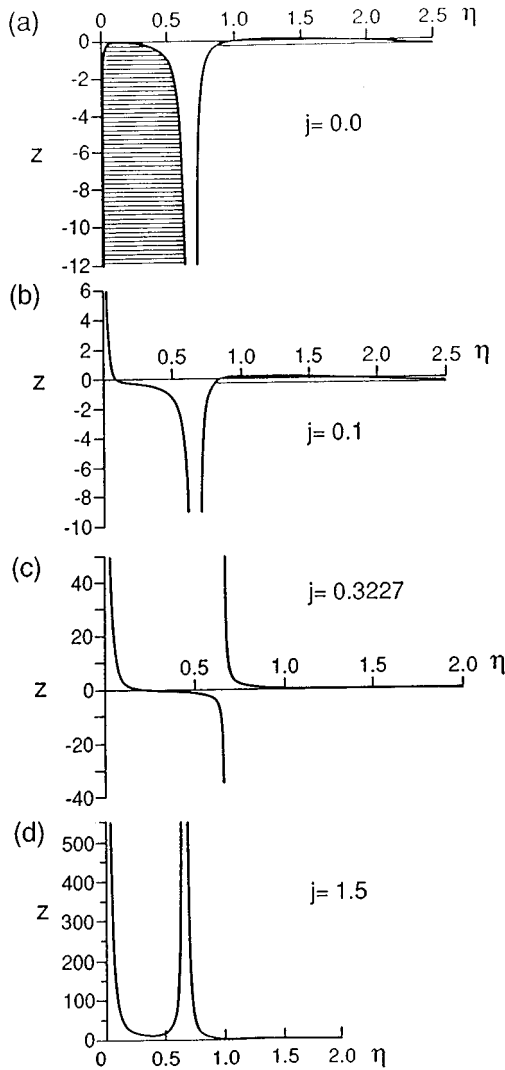


FIG. 5. Plots of  $Z = (d\eta/d\xi)^2$  as a function of  $\eta$  illustrating solutions of Eq. (5.26). Horizontal lines correspond to nonsingular periodic solutions as the integration constant shifts the coordinate system upwards. Here, we have used the parameter values of  $\alpha = 0.5$ ,  $\beta = 0.75$ ,  $\gamma = 0.25$ , and  $\delta = 1.0$ .

identical manner to that presented earlier [see Eq. (5.19) and thereafter]. We readily conclude that localized solutions to Eq. (5.30) exist whenever

$$\frac{E}{\omega_2} < \alpha^2 \left( 1 - \frac{\lambda\Lambda}{2} \right) + \frac{\lambda}{2}. \quad (5.32)$$

This imposes an *upper* bound on the localized superconducting state energy and also gives a possible condition for the destruction of superconductivity. The latter occurs when the phase gradient,  $\alpha$ , is large enough so that

$$(\nabla\phi)^2 = - \frac{\lambda}{2 \left( 1 - \frac{\lambda\Lambda}{2} \right)} \quad (5.33)$$

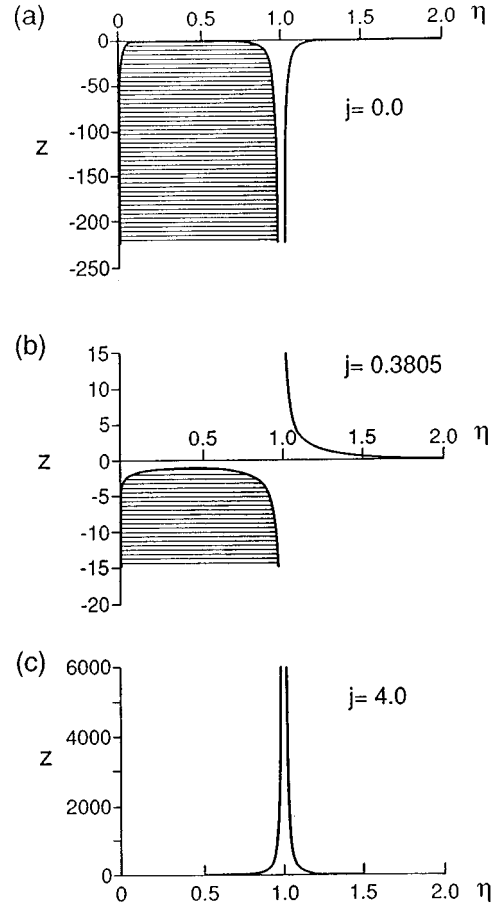


FIG. 6. Same as in Fig. 5 except the parameters are chosen as  $\alpha = 0.5$ ,  $\beta = \frac{1}{3}$ ,  $\gamma = 0.25$ , and  $\delta = 1.0$ .

and requires that  $\lambda\Lambda > 2$ , or in terms of the coupling constant, that  $|\eta_0| > \sqrt{\Omega_2 q_c^2}$ , i.e., the localized superconducting state is destroyed when electron-phonon coupling is strong enough. It does *not* necessarily mean that superconductivity disappears altogether but that spatial localization may no longer take place. This may be a possible criterion for type-II (inhomogeneous) versus type-I (homogeneous) superconductivity. Obviously, further studies are required to elucidate this question. The truly three-dimensional solutions may be obtained within this procedure and for technical hints on how this may be done, the reader is referred to Ref. [36].

A final comment we wish to make in this section regards the role of temperature in destroying superconductivity. The method we employed is a zero-temperature formalism and, as such, cannot be used to calculate the critical temperature. However, with upper bound restrictions on the energy of the superconducting state whether localized or spatially extended, the necessary condition on its existence is that close to  $T = T_c$  the energy of thermal fluctuations per degree of freedom  $kT_c$  is comparable with the energy  $E$  which appears in Eq. (5.32).

## VI. CONCLUSIONS

In this paper we have provided a method of analyzing coupled many-body systems of two degrees of freedom

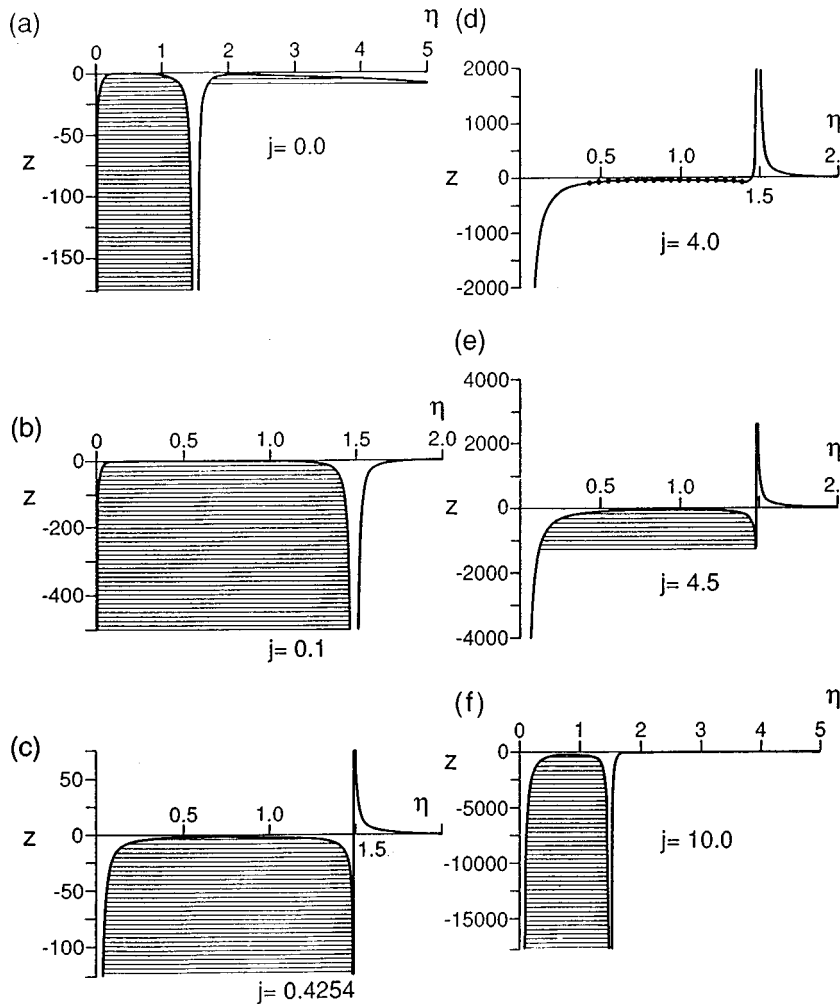


FIG. 7. Same as in Fig. 5 except the parameters are chosen as  $\alpha=0.5$ ,  $\beta=0.15$ ,  $\gamma=0.25$ , and  $\delta=1.0$ . Also note that the dotted line segment in (d) represents a localized nonsingular solution (solitary wave).

(critical or noncritical) which occur very frequently in condensed matter physics and also other branches of physics relying on many-body formalism. The approach to the problem is an important application to a very general class of problems using the method of coherent structures [9,10]. Since a growing number of important physical phenomena (superconductivity, metamagnetism, structural phase transitions, Mott insulators, etc.) can be adequately described only in terms of two or more interacting degrees of freedom, which lead to the formation of emergent phenomena such as superconductivity, we feel that our work represents a significant step forward in unifying these diverse systems within a single theory. In our paper, a general calculation of the field equations has been performed that can be applied to an arbitrary system with two quantized degrees of freedom provided its energy dispersion relations and coupling constants are known or can be estimated. All four types of two-component systems, i.e., boson-boson, fermion-fermion (distinguishable as well as indistinguishable), and fermion-boson have been investigated and their field equations of motion derived. We have also provided the reader with a brief overview of mathematical methods required for the solution of the coupled nonlinear differential equations that result. These approaches lead directly to a quantitative physical description of the behavior of a coupled field system. For example, it is now possible to locate chaotic regimes in the appropriate phase space and a knowledge of localized solutions provides

a description of particle densities as a function of physical parameters. The nature of the oscillations about stable equilibria is another significant outcome of the calculations. Classical solutions to the nonlinear coupled equations makes it possible to quantize and investigate the spectrum of associated quantum states.

In addition, we have provided two worked application sections. The first one, being the simplest, deals with the distinguishable fermion-fermion system (e.g., electron-proton plasmas) and demonstrates a direct connection between the equations of motion for the fields and the celebrated nonlinear Schrödinger equation leading to a wealth of analytical solutions including solitons.

In the second application section of this paper we have given a detailed analysis of the Fröhlich Hamiltonian for electron-phonon interactions in a metal which was so crucial in building a modern theory of superconductivity. Our approach has been fully nonlinear and analytical solutions have been found for both the phonon and electron fields under the general conditions of arbitrary dimensionality and the strength of coupling. The conclusion reached here is that in order to demonstrate Cooper pair formation in a Fröhlich-like Hamiltonian the electron-phonon coupling coefficient must be sufficiently strong for  $q$ -independent interactions or must not exceed a critical value for  $q$ -dependent couplings. Precise formulas linking the model parameters have been

obtained as quantitative criteria for superconductivity. If they are not satisfied, the only solutions which arise are either identically zero or singular, both of which exclude the existence of superconductivity.

We intend to carry out further applications to this and other important systems (structurally unstable crystal lattices,

Jahn-Teller compounds, magnetoelastically coupled spin systems, etc.) in a future work.

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